

# AN ELEGANT 3-BASIS FOR INVERSE SEMIGROUPS

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ABSTRACT. It is well known that in every inverse semigroup the binary operation and the unary operation of inversion satisfy the following three identities:

$$x = (xx')x \quad (xx')(y'y) = (y'y)(xx') \quad (xy)z = x(yz'').$$

The goal of this note is to prove the converse, that is, we prove that an algebra of type  $\langle 2, 1 \rangle$  satisfying these three identities is an inverse semigroup and the unary operation coincides with the usual inversion on such semigroups.

## 1. INTRODUCTION

In the language of a binary operation  $\cdot$  and a unary operation  $'$ , a set of  $n$  independent identities is an  $n$ -basis for inverse semigroups, if those identities define the variety of inverse semigroups considered as algebras  $(S, \cdot, ')$  of type  $\langle 2, 1 \rangle$ , where the unary operation coincides with the natural inversion. Denoting by  $x'$  the inverse of an element  $x$  in an inverse semigroup, we then have  $x = (xx')x$  (as inverse semigroups are regular semigroups) and  $(xx')(y'y) = (y'y)(xx')$  (as both  $xx'$  and  $y'y$  are idempotents, and idempotents commute in inverse semigroups). Thus we might be tempted to think that the following identities provide a 3-basis for inverse semigroups:

$$x = (xx')x, \quad (xx')(y'y) = (y'y)(xx') \quad \text{and} \quad (xy)z = x(yz). \quad (1.1)$$

However, for  $S = \{0, 1\}$  with  $xy = 0$ , except for  $11 = 1$ , and defining  $x' = 1$ , we have the previous identities satisfied, but  $0' \neq 0'00'$  and hence  $'$  does not coincide with the natural inversion in  $(S, \cdot)$ .

B.M. Schein [4] repaired the *defect* of (1.1) by adjoining two additional identities:  $x'' = x$  and  $(xy)' = y'x'$ . The resulting set of five identities indeed provides a 4-basis for inverse semigroups. (The identity  $(xy)' = y'x'$  is dependent upon the others, and hence can be discarded. However it is worth observing that in the same paper Schein also provided a 5-basis using  $xx'x'x = x'xxx'$  instead of  $xx'y'y = y'yyx'$ ; see [4, Theorem 1.6] and [2, p. 15, Ex. 20(b)].) Therefore the natural question to ask would be: *is it possible to find a 3-basis for inverse semigroups?* This question was first answered in the affirmative in [1], but the 3-basis given there requires an extremely complicated proof (it is still an open problem to provide a reasonable proof for that result).

The aim of this note is to repair (1.1) by providing an easy, transparent and *elegant* 3-basis for inverse semigroups.

**Theorem.** *Let  $(S, *, ')$  be an algebra of type  $\langle 2, 1 \rangle$ . Then this algebra is an inverse semigroup and the unary operation coincides with the usual inversion on such semigroups if and only if*

$$(\mathbf{E}_1) \quad x = (xx')x, \quad (\mathbf{E}_2) \quad (xx')(y'y) = (y'y)(xx'), \quad (\mathbf{E}_3) \quad (xy)z = x(yz'').$$

## 2. PROOF OF THE THEOREM

In this section we prove that the identities  $(\mathbf{E}_1)$ – $(\mathbf{E}_3)$  imply Schein's 4-basis for inverse semigroups. As the converse is obvious, the equivalence of the two bases will follow.

Throughout this section let  $(S, \cdot, ')$  be an algebra of type  $\langle 2, 1 \rangle$  satisfying  $(\mathbf{E}_1)$ – $(\mathbf{E}_3)$ . We start by proving a few handy identities.

**Lemma 1.** *The following identities hold.*

$$x'x'' = x'x \quad (2.1)$$

$$(xy')y = x(y'y) \quad (2.2)$$

$$x = x(x'x) \quad (2.3)$$

$$x'' = (x''x')x = x''(x'x) \quad (2.4)$$

$$x'''x = x'''x'' = x'''x^{(4)} \quad (2.5)$$

*Proof.* Firstly, for (2.1), we have

$$x'x'' \stackrel{(\mathbf{E}_1)}{=} x'[(x''x''')x''] \stackrel{(\mathbf{E}_3)}{=} [x'(x''x''')]x \stackrel{(\mathbf{E}_3)}{=} [(x'x'')x']x \stackrel{(\mathbf{E}_1)}{=} x'x.$$

Next, for (2.2), we compute  $(xy')y \stackrel{(\mathbf{E}_3)}{=} x(y'y'') \stackrel{(2.1)}{=} x(y'y)$ .

Regarding (2.3), we have  $x(x'x) \stackrel{(2.2)}{=} (xx')x \stackrel{(\mathbf{E}_1)}{=} x$ .

Then for (2.4), we compute  $x'' \stackrel{(2.3)}{=} x''(x'''x'') \stackrel{(\mathbf{E}_3)}{=} (x''x''')x \stackrel{(2.1)}{=} (x''x')x \stackrel{(2.2)}{=} x''(x'x)$ .

Finally, for (2.5), we have

$$x'''x \stackrel{(2.3)}{=} [x'''(x''''x''')]x \stackrel{(\mathbf{E}_3)}{=} x'''[(x''''x''')x''] \stackrel{(2.4)}{=} x'''x'''' \stackrel{(2.1)}{=} x'''x''.$$

□

The next two lemmas are the key tools in the proof that the identities  $(\mathbf{E}_1)$ – $(\mathbf{E}_3)$  imply  $x'' = x$ .

**Lemma 2.**  $(x'x)x''' = x'''$ .

*Proof.* We start with two observations. Firstly, as

$$[x(y'''y)]y' \stackrel{(\mathbf{E}_3)}{=} x[(y'''y)y'''] \stackrel{(2.5)}{=} x[(y'''y''')y'''] \stackrel{(\mathbf{E}_1)}{=} xy''',$$

we have

$$(x(y'''y))y' = xy'''. \quad (2.6)$$

Secondly,

$$(x'x)(x'''x) \stackrel{(2.5)}{=} (x'x)(x'''x''') \stackrel{(\mathbf{E}_2)}{=} (x'''x''')(x'x) \stackrel{(2.5)}{=} (x'''x'')(x'x) \stackrel{(2.2)}{=} [(x'''x'')x']x \stackrel{(2.4)}{=} x'''x,$$

so that

$$(x'x)(x'''x) = x'''x. \quad (2.7)$$

Now we have all we need to prove the lemma.

$$x''' \stackrel{(2.4)}{=} (x'''x'')x' \stackrel{(2.5)}{=} (x'''x)x' \stackrel{(2.7)}{=} [(x'x)(x'''x)]x' \stackrel{(2.6)}{=} (x'x)x'''.$$

□

**Lemma 3.**  $(xy)z' = x(yz')$ .

*Proof.* We start by proving that

$$x''' = x'. \quad (2.8)$$

In fact we have  $xx' \stackrel{(2.3)}{=} [x(x'x)]x' \stackrel{(\mathbf{E}_3)}{=} x[(x'x)x'''] = xx'''$ , using Lemma 2 in the last equality. Thus

$$xx''' = xx'. \quad (2.9)$$

Now, by Lemma 2,

$$x''' = (x'x)x''' \stackrel{(2.1)}{=} (x'x'')x''' \stackrel{(\mathbf{E}_3)}{=} x'(x''x^{(5)}) \stackrel{(2.9)}{=} x'(x''x''') \stackrel{(\mathbf{E}_3)}{=} (x'x'')x' \stackrel{(\mathbf{E}_1)}{=} x'.$$

Replacing  $z$  by  $z'$  in  $(\mathbf{E}_3)$ , we get

$$(xy)z' = x(yz''') = x(yz'),$$

where the last equality follows from (2.8). The lemma is proved.  $\square$

We have everything we need to prove our main result.

**Theorem 1.** *The identities  $(\mathbf{E}_1)$ – $(\mathbf{E}_3)$  imply  $x'' = x$  and the associative law.*

*Proof.* First, we have

$$\begin{aligned} x''x' &\stackrel{(2.4)}{=} [(x''x')x]x' = (x''x')(xx') \stackrel{(\mathbf{E}_2)}{=} (xx')(x''x') \\ &= [(xx')x'']x' = [x(x'x'')]x' = x[(x'x'')x'] \stackrel{(\mathbf{E}_1)}{=} xx', \end{aligned}$$

where we have used Lemma 3 in the unlabeled equalities. Thus

$$x''x' = xx'. \quad (2.10)$$

Now  $x'' \stackrel{(2.4)}{=} (x''x')x \stackrel{(2.10)}{=} (xx')x \stackrel{(\mathbf{E}_1)}{=} x$ , as claimed.

Associativity now follows easily:  $(xy)z \stackrel{(\mathbf{E}_1)}{=} x(yz'') = x(yz)$ .  $\square$

### 3. OTHER SETS OF AXIOMS

It is natural to ask how sensitive the axioms  $(\mathbf{E}_1)$ – $(\mathbf{E}_3)$  are to certain modifications, such as shifting the parentheses in  $(\mathbf{E}_1)$  or changing the placement of the double inverse in  $(\mathbf{E}_3)$ .

If, for instance, we leave  $(\mathbf{E}_2)$  intact, replace  $(\mathbf{E}_1)$  with  $x(x'x) = x$  and replace  $(\mathbf{E}_3)$  with  $(x''y)z = x(yz)$ , then we obtain a set of identities which are dual to  $(\mathbf{E}_1)$ – $(\mathbf{E}_3)$ . By an argument dual to that in §2, this set of identities is another 3-basis for inverse semigroups.

Thus to dispense with these sorts of obvious dualities, we will assume that both  $(\mathbf{E}_1)$  and  $(\mathbf{E}_2)$  are left intact, and consider only alternative placement of the double inverse in  $(\mathbf{E}_3)$ . Using PROVER9, we found that each of the following identities can substitute for  $(\mathbf{E}_3)$  to give another 3-basis for inverse semigroups:

$$\begin{aligned} (xy)z &= x''(yz) & (xy)z &= x(y''z) \\ x(yz) &= (xy'')z & x(yz) &= (xy)z''. \end{aligned}$$

The remaining possibility,  $x(yz) = (x''y)z$ , does not work. Using MACE4, we found the counterexample given by the following tables. It satisfies  $(\mathbf{E}_1)$ ,  $(\mathbf{E}_2)$  and  $x(yz) = (x''y)z$ , but the binary operation is not associative ( $((0 \cdot 0) \cdot 0 = 1 \cdot 0 = 7 \neq 6 = 0 \cdot 1 = 0 \cdot (0 \cdot 0))$ ), and the unary operation clearly fails to satisfy  $x'' = x$ .

·	0	1	2	3	4	5	6	7	8	9	10	11
0	1	6	5	7	3	8	4	2	0	4	4	4
1	7	2	6	0	8	4	5	1	3	5	5	5
2	5	8	3	6	1	7	0	4	2	0	0	0
3	8	0	7	4	6	2	1	3	5	1	1	1
4	3	7	1	8	5	6	2	0	4	2	2	2
5	6	4	8	2	7	0	3	5	1	3	3	3
6	0	1	2	3	4	5	6	7	8	6	6	6
7	4	3	0	5	2	1	7	8	6	7	7	7
8	2	5	4	1	0	3	8	6	7	8	8	8
9	0	1	2	3	4	5	6	7	8	9	10	6
10	0	1	2	3	4	5	6	7	8	10	9	6
11	0	1	2	3	4	5	6	7	8	6	6	11
'	0	1	2	3	4	5	6	7	8	9	10	11
	1	2	3	4	5	0	6	8	7	9	10	11

#### 4. PROBLEM

*Does there exist a 2-basis for inverse semigroups?*

We guess that the answer is no.

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